PHYSICAL REVIEW E VOLUME 56, NUMBER 1 JULY 1997

Lyapunov exponents and coalescence of chaotic trajectories

L. Longa,[1,2] S. P. Dias,[2,3] and E. M. F. Curado[2,4,*]
[1]Department of Statistical Physics, Jagiellonian University, Reymonta 4, Kraków, Poland
[2]International Centre of Condensed Matter Physics, Universidade de Brasília, 70919-970 Brasília, DF, Brazil
[3]Instituto de Física, Universidade Federal Fluminense, Niterói, RJ, Brazil
[4]Centro Brasileiro de Pesquisas Físicas, 22290-180, Rio de Janeiro, RJ, Brazil

(Received 16 December 1996)

Identical nonlinear chaotic systems linked by a common noise term (or signal) may synchronize. The synchronization process, which is a combined effect of the noise and of the deterministic part of the map, could show much more complex behavior than the one suggested by recent studies. In particular, it is demonstrated that when the noise couples the states of an ensemble of identical systems the change of sign of the largest Lyapunov exponent of the ensemble is not necessarily connected with the synchronization. [S1063-651X(97)06107-2]

PACS number(s): 05.45.+b, 05.40.+j

Time evolution of nonlinear deterministic systems may exhibit extreme sensitivity to initial conditions, also referred to as chaotic behavior [1]. In practice it means that the observed trajectories display a random character making long-time evolution of the systems unpredictable [1]. Consequently, identical chaotic systems that start their evolution from different initial points in phase space are not expected to synchronize, and adding random noise should make them even ''more random.''

Recently, a number of counterintuitive examples to the above expectations has been found [2–6]. Namely, these papers have demonstrated that copies of a chaotic system, each evolving with *different* initial conditions, may synchronize under an identical sequence of a signal or a random force. That is, if we take a chaotic map and start numerical evolution of two arbitrarily chosen initial points subject to the same sequence of noise, the resulting trajectories will collapse to a single random trajectory after a finite number of iterations.

In our recent paper [7] we showed that the observation of coalescence should be atributed to finite precision of the calculations. Strictly speaking, in the majority of the cases studied in the literature the average coalescence time is either linear or an exponential function of precision and, statistically, coalescence never occurs when precision is infinite. The linear case, studied by Pecora and Carroll and Fahy and Hamann [2] (PF) differs, however, from the exponential one of Maritan and Banavar [4]. Namely, most of the processes with the average coalescence time being a linear function of precision cause the average distance between the random trajectories to converge to zero with the number of iterations. It is this case by which, here and in the rest of this paper, we mean the coalescence (synchronization) of random trajectories. More specifically, *we shall consider random trajectories as synchronized (collapsed) if the corresponding average distance between them, converges to zero with time.*

*Also at Departamento de Física, Universidade de Brasília, DF, Brazil.

One may wonder as to whether synchronized trajectories can be distinguished from nonsynchronized ones by studying the sign of the maximal Lyapunov exponent. Such correlation was suggested by Pikovsky and Kaulakys, Ivanauskas, and Meskauskas [3,6] and by the PF models where synchronization took place when the largest Lyapunov exponent became negative. Indeed, for the identical one-dimensional chaotic systems subject to random forces generated independently of the states of the systems [2,3], the change of sign of the largest Lyapunov exponent from positive to negative values implied going from a nonsynchronized to a synchronized regime. In the cases studied by Pikovski [3] the largest Lyapunov exponent of the ensemble was found by studying a single subsystem [3].

A purpose of this paper is to show that the lack of synchronization does not necessary imply that the maximal Lyapunov exponent must be positive. For the logistic map, when the noise couples with the states of the systems (as in the case of Maritan and Banavar model (MB) [4]) the technique proposed by Pikovsky [3] does not apply and the difference between ensembles with positive and negative Lyapunov exponents becomes nontrival. We demonstrate that in this case the Lyapunov exponents are not the most useful characterization of the synchronization process. The situation resembles the complexity of dynamics observed for chaotic maps subject to a random perturbation [8]. Indeed, as we shall prove, the models of collapse studied by MB belong to this category.

In order to clarify a relation between the coalescence and the Lyapunov exponents we shall consider a generalization of the MB model. They studied chaotic logistic maps [1,9–13] coupled by an external noise

$$\begin{aligned} x' &= 4\,x(1-x) + W\eta, \\ y' &= 4\,y(1-y) + W\eta, \end{aligned} \tag{1}$$

where $0 \leqslant x,y \leqslant 1$; η is a (common for both subsystems) random number chosen uniformly from the interval -1 to $+1$ and $W>0$ is the strength of the noise. The values of η violating the bounds $0 \leqslant x',y' \leqslant 1$ is discarded and a new η is

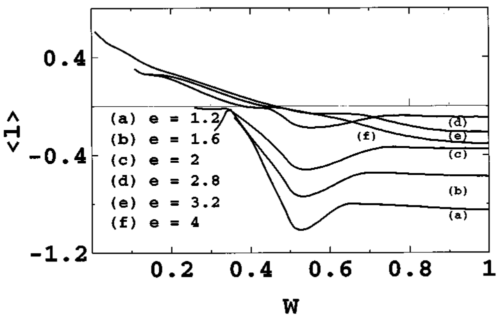

FIG. 1. Lyapunov exponents as functions of W for the $W-\Omega_e$ process.

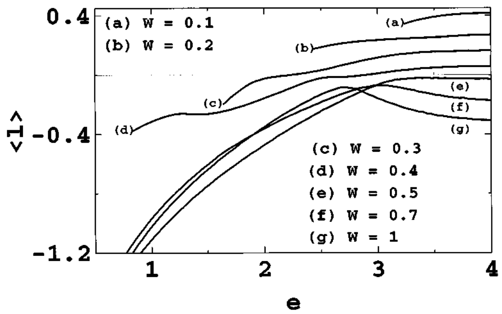

FIG. 2. Lyapunov exponents as functions of e for the $W-\Omega_e$ process with fixed W.

chosen. Though this model does not show synchronization [7] it provides a simple example of the noise induced coupling between the initially uncorrelated systems ($W=0$). A straightforward generalization of the process (1) allows one to study the coalescence process in a more detailed way. In order to introduce the generalized model let us note that chaotic maps have iterations that are composed of two parts, one which stretches the distance between the points and another where the distance is enlarged. Consider the distance function d_l between images of x and y for process (1)

$$d_l : \mathcal{S} \ni (x,y) \rightarrow 4|x-y||1-(x+y)|, \tag{2}$$

where $\mathcal{S}=[0,1]\times[0,1]$. Then the distance contracting area Ω_1, of map (1) is given by the condition: $\{(x,y)$: $3/4<x+y<5/4\}\cap\mathcal{S}$. That is, $\forall$ $(x,y)\in\Omega_1$: $d_l(x,y)<|x-y|$. Clearly, for points in $\mathcal{S}$ but outside Ω_1 the distance between the points is either enlarged or left unchanged.

Let us now introduce such a generalization of the random chaotic map (1) that allows us to *control* the values of $d_l(x,y)$. For that purpose let the states (x,y) be restricted to an area $\Omega_e \subset \mathcal{S}$, given by the inequality $1-e/4 \leqslant x+y \leqslant 1+e/4$. In particular, $\Omega_4=\mathcal{S}$. Suppose the evolution proceeds, as previously, according to Eq. (1), but with $(x',y')\in\Omega_e$. This condition is realized in practice by an appropriate choice of the noise term. Namely, again η is a (common for both subsystems) random number chosen uniformly from the interval -1 to $+1$. In order to guarantee that the noise transfers all points from $\mathcal{S}$ to Ω_e we require that $W>\frac{1}{2}(1-e/4)$.

The values of η yielding (x',y') from outside Ω_e are discarded and a new η is chosen. That is, the noise term "shuts" the points $0\leqslant 4x(1-x),4y(1-y)\leqslant 1$ (obtained from the previous iteration) into Ω_e. We shall call this process $W-\Omega_e$. Clearly, for the whole process $d_l(x,y)\leqslant e|x-y|$ and for $e=4$ we recover the model proposed by MB. One of the interesting features of the generalized model is that for $e\leqslant 1$ $(W>\frac{1}{2}(1-e/4))$ it always contracts the distance between the points so it certainly yields a collapse of the trajectories. Actually, as we demonstrate, the collapse takes place for a much larger set of parameters. For example, when $W\geqslant 1$ the collapse is observed for $e\lesssim 2.717$. The regions $e\lessgtr 2.717$ are separated by a parametric second order phase transition, where the average coalescence time as function of precision changes from algebraic to exponential dependency. The averaged distance $<d_l>$ plays the role of an *order parameter*.

A study of the phenomena of synchronization is simplified if we note that the $W-\Omega_e$ processes can be written in an equivalent form as

$$\begin{aligned} x' &= X_M(\tilde{x},\tilde{y}) + \eta\tilde{W}(\tilde{x},\tilde{y},W),\\ y' &= Y_M(\tilde{x},\tilde{y}) + \eta\tilde{W}(\tilde{x},\tilde{y},W),\\ \tilde{x} &= 4x(1-x), \quad \tilde{y}=4y(1-y). \end{aligned} \tag{3}$$

Here η is an independent random variable, extracted *at each time step* with uniform distribution on the interval $[-1,1]$ (and without constrains). The functions X_M, Y_M, and $\tilde{W}$ read $(W>\frac{1}{2}(1-e/4))$

$$X_M=\frac{T+B}{2}, \quad Y_M=X_M+\tilde{y}-\tilde{x}, \quad \tilde{W}=\frac{T-B}{2} \tag{4}$$

where

$$T=\min\left\{1,1-(\tilde{y}-\tilde{x}),\frac{1}{2}\left[1+\frac{e}{4}-(\tilde{y}-\tilde{x})\right],\tilde{x}+W\right\} \tag{5}$$

$$B=\max\left\{0,-(\tilde{y}-\tilde{x}),\frac{1}{2}\left[1-\frac{e}{4}-(\tilde{y}-\tilde{x})\right],\tilde{x}-W\right\}. \tag{6}$$

The largest Lyapunov exponent $\langle l\rangle$ for process (3) is found in a standard way by calculating the Jacobian $\boldsymbol{J_i}$ at a time i and by using the relation

$$\begin{aligned} \langle l\rangle &= \lim_{K\to\infty}\frac{1}{K}\ln|\boldsymbol{J}_K\boldsymbol{J}_{K-1}\cdots\boldsymbol{J}_2\boldsymbol{J}_1\hat{\boldsymbol{v}}_0|\\ &= \lim_{K\to\infty}\frac{1}{K}(\ln|\boldsymbol{v}_K|+\ln|\boldsymbol{v}_{K-1}|+\cdots+\ln|\boldsymbol{v}_1|), \end{aligned} \tag{7}$$

where $\hat{\boldsymbol{v}}_0$ is an arbitrary two-dimensional vector of unit length $(|\hat{\boldsymbol{v}}_0|=1)$ and where

$$|\boldsymbol{v}_{i+1}|=|\boldsymbol{J}_{i+1}\hat{\boldsymbol{v}}_i| \quad \hat{\boldsymbol{v}}_i=\frac{\boldsymbol{v}_i}{|\boldsymbol{v}_i|}. \tag{8}$$

Results are shown in Figs. 1 and 2. Note that for the $W-\Omega_e$ process there is no correlation between the change of sign of $\langle l\rangle$ and the collapse. Consider, for example, two special cases: (a) the original MB process with $e=4$ and $0\leqslant W\leqslant 1$ and (b) the $W-\Omega_e$ process with $W=1$. For the MB process the collapse does not take place [7] but, nonetheless, $\langle l\rangle$ changes sign as a function of W. On the other

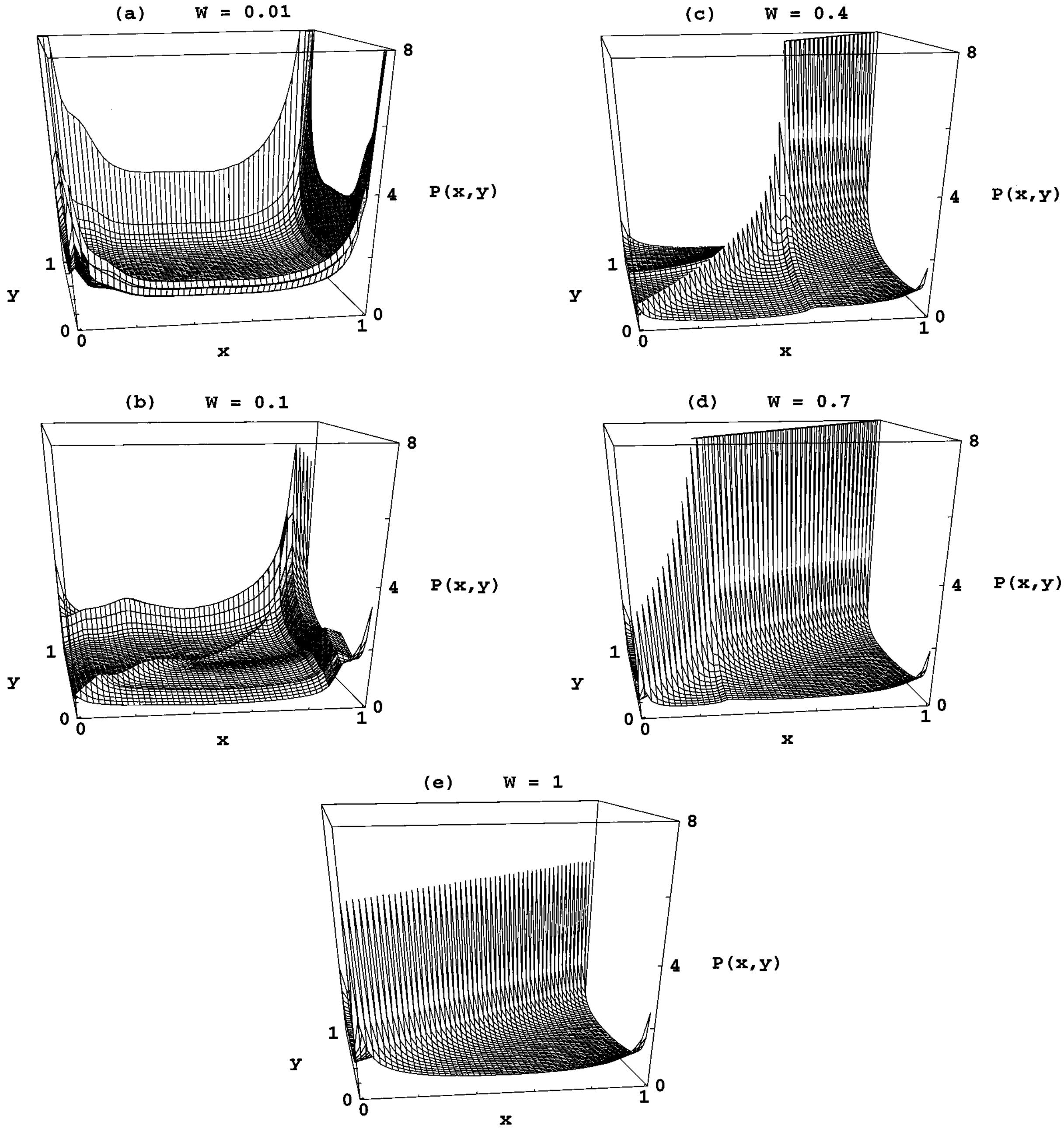

FIG. 3. Pair distribution function for the $W-\Omega_4$ process and for various parameters W: (a) $W=0.01$, (b) $W=0.1$, (c) $W=0.4$, (d) $W=0.7$, and (e) $W=1$.

hand, for the $W-\Omega_e$ process the largest Lyapunov exponent is always negative although collapse takes place only for $e\approx 2.717$. *For* $e>2.717$ *the collapse does not occur in spite of fact that the Lyapunov exponent is negative.*

In order to identify relevant factors responsible for the value of the maximal Lyapunov exponent, we have calculated $\langle l\rangle$ for cases (a) and (b) by a direct diagonalization of the product $\boldsymbol{J}_K\boldsymbol{J}_{K-1}\cdots\boldsymbol{J}_2\boldsymbol{J}_1$. For these particular cases this diagonalization can be done exactly, yielding

$$\langle l\rangle=\lim_{K\to\infty}\frac{1}{K}\left\{\sum_{n=1}^{K}\ln\left|4-8y_n+4(y_n-x_n)[\tilde{T}_{n-1}+\tilde{B}_{n-1}+\eta_{n-1}(\tilde{T}_{n-1}-\tilde{B}_{n-1})]\right|\right\}, \tag{9}$$

where

$$\tilde{T}_m=\Theta(\tilde{y}_m-\tilde{x}_m)\Theta\left(\frac{e}{4}-1+\tilde{y}_m-\tilde{x}_m\right)+\frac{1}{2}\Theta\left(1-\frac{e}{4}+\tilde{y}_m-\tilde{x}_m\right)\Theta\left(1-\frac{e}{4}-\tilde{y}_m+\tilde{x}_m\right), \tag{10}$$

$$\tilde{B}_m=\Theta(\tilde{x}_m-\tilde{y}_m)\Theta\left(\frac{e}{4}-1-\tilde{y}_m+\tilde{x}_m\right)+\frac{1}{2}\Theta\left(1-\frac{e}{4}-\tilde{y}_m+\tilde{x}_m\right)\Theta\left(1-\frac{e}{4}+\tilde{y}_m-\tilde{x}_m\right) \tag{11}$$

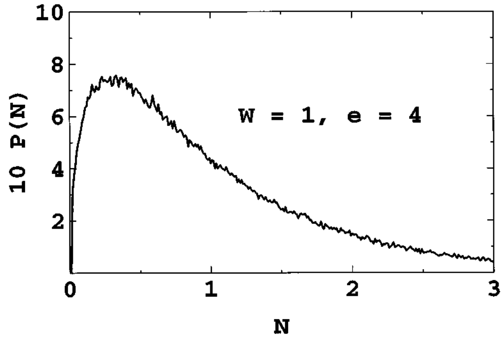

FIG. 4. The distribution function $P(N)$ that N digits of x and y are identical ($N=|\ln 10(|x-y|)|$). Calculations are carried out for the $W-\Omega_4$ process with $W=1$.

and where Θ is the θ Heaviside. Additionally $\widetilde{T}_0 \equiv \widetilde{T}_K$ and $\widetilde{B}_0 \equiv \widetilde{B}_K$. Note that Eq. (9) depends only on the properties of the pair distribution function $P(x,y)$. This follows from the observation that $\widetilde{x}_n$ and $\widetilde{y}_n$ can be expressed in terms of $\widetilde{x}_{n-1}$, $\widetilde{y}_{n-1}$, and η_{n-1}. Furthermore, the averaging over the independent uniformly distributed random numbers η_{n-1} can be done separately. For $W=1$ and for $e=4$ the final formulas becomes particularly simple. It reads

$$\langle l \rangle = \ln(4) - 1 + \lim_{K\to\infty} \frac{1}{K} \sum_{n=1}^{K} \ln|1 - 8|x_n - y_n||1 - x_n - y_n||, \tag{12}$$

$$= \ln(4) - 1 + \int_0^1 \int_0^1 dx\, dy\, P(x,y) \times \ln|1 - 8|x-y||1-x-y||. \tag{13}$$

From Eq. (13) it becomes clear that the sign of the maximal Lyapunov exponent for the MB process depends on details of *pair correlations* and *even for trajectories that do not collapse it can be negative*. To see this explicitely we sketched in Fig. 3 the evolution of the pair correlation function $P(x,y)$ with increasing W. This function fulfills a stochastic Frebonius-Peron integral equation, similar to that given in [8], and can easily be generated numerically. Note that the $W-\Omega_4$ process for small W's is practically equivalent to the dynamics of the two decoupled logistic maps. Also, the corresponding Lyapunov exponent is similar to that known for the logistic map. With increasing W the noise

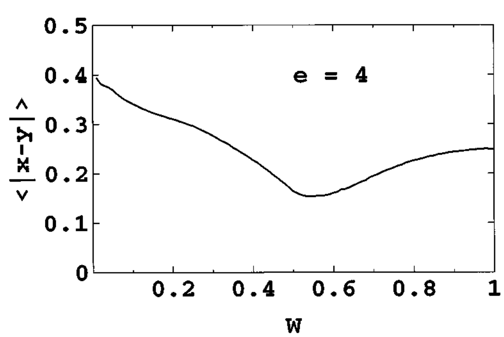

FIG. 5. The distance function $\langle d_l \rangle \equiv \langle |\boldsymbol{x}-\boldsymbol{y}| \rangle$ for the $W-\Omega_4$ process.

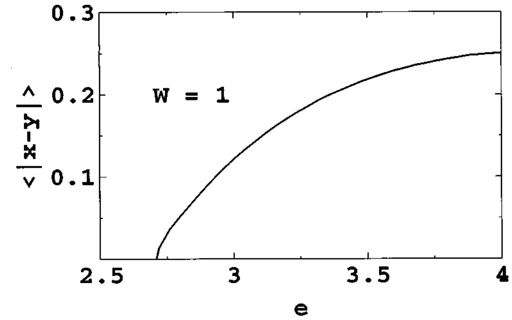

FIG. 6. The distance function $\langle d_l \rangle \equiv \langle |\boldsymbol{x}-\boldsymbol{y}| \rangle$ for the $W-\Omega_e$ process with $W=1$. It shows a parametric phase transition for $e \approx 2.717$.

correlates subsystems. These correlations are already seen for $W=0.1$ in the form of a shallow local maximum along the $x=y$ line and become fully developed for $W \geqslant 0.4$. Hence, the noise term correlates subsystems and the sign of $\langle l \rangle$ is determined by details of $P(x,y)$. Neither very strong correlations $[P(x,y)=\delta(x-y)$, where δ is the Dirac function$]$ nor lack of correlations $[P(x,y)=P(x)P(y)]$ would yield negative values of $\langle l \rangle$. The fine structure of the pair correlations, giving a probability distribution that the two trajectories have N common digits, is shown in Fig. 4. We clearly see that for $W=1$ no collapse occurs.

From the analysis above it seems that Lyapunov exponents do not provide the most general characterization of the collapse processes and some other quantities should be looked for. The most natural and almost trivial one is the distance function

$$\langle d_l \rangle = \lim_{K\to\infty} \frac{1}{K} \sum_{n=1}^{K} \sqrt{\sum_{\alpha} (x_{\alpha,n} - y_{\alpha,n})^2}, \tag{14}$$

where x_α and y_α are the components of a multidimensional vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ describing a chaotic, random process. For the $W-\Omega_e$ processes this function has been shown in Figs. 5 and 6. Note that the $\langle d_l \rangle$ serves, in this case, as an order parameter, i.e., it vanishes for infinite times in the collapse area and is nonzero otherwise. For the $W-\Omega_e$ process ($W=1$) we observe a second order phase transition at $e=e_c$ ($e_c \approx 2.717$) from a noncollapsing regime ($e > e_c$) to a collapsing one ($e < e_c$). Close to the phase transition $\langle |x-y| \rangle \approx (e-e_c)^\beta$, where β is a critical exponent. The estimates give $\beta = 1 \pm 0.25$. Both, e_c and β are very hard to get. They were found with the help of a program running under control of the MAPLE package. Averages were performed over 10^6 time steps with an adaptive precision scheme which guaranteed that statistically no collapse took place within the first 10^6 time steps. Close to e_c a relative precision of 1536 digits was necessary to fulfill this condition. But still the data are characterized by a large scatter making direct estimates of β extremely difficult.

ACKNOWLEDGMENTS

This work was supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and by the Financiadora de Estudos e Projetos (FINEP) in Brazil.

[1] See, e.g., (a) H. G. Schuster, *Deterministic Chaos: An Introduction* (VCH, Verlagsgesellschaft, Germany, 1989); (b) J. Ford, in *Directions in Chaos*, edited by Hao Bai-Lim (World Scientific, Singapore, 1989), Vol. 3.
[2] (a) L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990); (b) S. Fahy and D. R. Hamann, *ibid.* **69**, 761 (1992).
[3] A. S. Pikovsky, Phys. Lett. A **165**, 33 (1992).
[4] A. Maritan and J. R. Banavar, Phys. Rev. Lett. **72**, 1451 (1994); **73**, 2932 (1994).
[5] A. S. Pikovsky, Phys. Rev. Lett. **73**, 2931 (1994).
[6] B. Kaulakys, F. Ivanauskas, and T. Meskauskas, Phys. Rev. E **52**, 2091 (1995).
[7] L. Longa, E. M. F. Curado, and F. A. Oliveira, Phys. Rev. E **54**, R2201 (1996).
[8] G. Paladin, M. Serva, and A. Vulpiani, Phys. Rev. Lett. **74**, 66 (1995).
[9] (a) M. J. Feigenbaum, Los Alamos Sci. **1**, 4 (1980); (b) P. Coullet and J. Tresser, J. Phys. (France) Colloq. **39**, C5 (1978); (c) R. M. May, Nature (London) **261**, 459 (1976).
[10] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, England, 1993).
[11] C. Tsallis, A. M. C. de Souza, and E. M. F. Curado, Chaos Solitons Fractals **6**, 561 (1995).
[12] F. A. Oliveira, Phys. Rev. B **52**, 1009 (1995).
[13] C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. A **38**, 3688 (1988).